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A simple lattice version of the nonlinear Schrödinger equation and its deformation with an exact quantum solution

A Kundu†§ and O Ragnisco‡

† Physikalisches Institut Der Universität Bonn Nussallee 12, 53115 Bonn, Germany
‡ Università di Roma III, Dipartimento di Fisica and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, P le A Moro 2, 00185 Rome, Italy

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Abstract. A lattice version of the quantum nonlinear Schrödinger (NLS) equation is considered, which has a significant simple form and fulfils most of the criteria desirable for such lattice variants of field models. Unlike most of the known lattice NLS equations, the present model belongs to a class which does not exhibit the usual symmetry properties. However, this lack of symmetry itself seems to be responsible for the remarkable simplification of the relevant objects in the theory, such as the Lax operator, the Hamiltonian and other commuting conserved quantities as well as their spectra. The model allows exact quantum solution through the algebraic Bethe ansatz and also a straightforward and natural generalization to the vector case, thus giving a new exact lattice version of the vector NLS model. A deformation representing a new quantum integrable system involving Tamm–Dancoff-like q-boson operators is constructed.

1. Introduction

Among integrable systems, discrete models represent a special class, interest in which has been revived in recent years [1–3]. In the context of quantum integrable systems, apart from being themselves solvable quantum lattice models, they also play an important role by providing lattice regularized versions of the corresponding continuum models. Thus, the lattice nonlinear Schrödinger (NLS) [4–9] and lattice sine–Gordon [7] models, etc, are useful for finding out the exact quantum solutions of the related field models through the quantum inverse-scattering method (QISM) [10]. Moreover, these lattice versions are often able to unveil the hidden algebraic structures of the original field models [1, 11].

Ideally, a candidate for such quantum integrable discrete models represented by the Lax operator $L(n|\lambda)$ should fulfil the following basic criteria:

(i) It should satisfy exactly the quantum Yang–Baxter equation (QYBE)

$$R(\lambda-\mu)L(\lambda)\otimes L(\mu) = L(\mu)\otimes L(\lambda)R(\lambda-\mu). \tag{1.1}$$

(ii) It should have the same quantum R-matrix as the corresponding field model.

(iii) The discrete Lax operator should yield the continuum operator at $\Delta \to 0$: $L(n|\lambda) \to \mathbf{1} + \Delta\mathcal{L}(x,\lambda)$, Δ being the lattice constant.

(iv) The Hamiltonian of the discrete model should be 'local' and return the local field model at the continuum limit.

§ Permanent address: Theoretical Nuclear Physics Division, Saha Institute of Nuclear Physics, Sector I, Block-AF, Bidhannagar, Calcutta-700 064, India.

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Moreover, as a desirable physical requirement, *the Lax operator, as well as the Hamiltonian and the energy spectrum, should be as simple as possible.*

A discrete version of the NLS model was first suggested by Ablowitz and Ladik [5]. However, the corresponding R-matrix, a key object in the QISM, is expressed through trigonometric functions [12] and does not coincide with the well known rational R-matrix of the NLS field model, which reads as

$$R(\lambda)=\lambda P-\mathrm{i}\kappa I=\begin{pmatrix}\lambda-\mathrm{i}\kappa & & & \\ & -\mathrm{i}\kappa & \lambda & \\ & \lambda & -\mathrm{i}\kappa & \\ & & & \lambda-\mathrm{i}\kappa\end{pmatrix}. \tag{1.2}$$

That is, criterion (ii) laid down above is not satisfied. Subsequently, a different discrete NLS was proposed [6], constructed through the Holstein–Primakoff transformation (HPT) applied to an infinite-dimensional irreducible representation of $su(2)$ with the classical Lax operator given by

$$L(n|\lambda)=\begin{pmatrix} c-\frac{1}{2}\lambda\Delta+\frac{\kappa}{2}\Delta^2\psi(n)^*\psi(n) & -\mathrm{i}\psi(n)^*\Delta\sqrt{\kappa(c+\frac{\kappa}{4}\Delta^2\psi(n)^*\psi(n))} \\ \mathrm{i}\Delta\sqrt{\kappa(c+\frac{\kappa}{4}\Delta^2\psi(n)^*\psi(n))}\psi(n) & c+\frac{1}{2}\lambda\Delta+\frac{\kappa}{2}\Delta^2\psi(n)^*\psi(n)\end{pmatrix} \tag{1.3}$$

related to spin parameter $s=-\frac{2}{\kappa\Delta}$ with $c=1$ and canonical Poisson brackets $\{\psi(n),\psi(m)^*\}=\delta_{nm}$.

This model is free from the earlier drawback, namely it satisfies (ii), though it now fails to fulfil the locality criterion (iv) at the quantum level. As a remedy, another version of the lattice NLS was introduced [7], represented by a Lax operator which depends explicitly on lattice points and may be expressed as a product $l(n|\lambda)=L(2n|\lambda)L(2n-1|\lambda)$, where $L(n|\lambda)$ is taken as in (1.3) with $c=1+\frac{1}{4}(-1)^n\kappa\Delta$, i.e.

$$\begin{aligned}L(n|\lambda)&\\ =&\begin{pmatrix} 1+\frac{1}{4}(-1)^n\kappa\Delta-\frac{1}{2}\lambda\Delta+\frac{\kappa}{2}\Delta^2\psi(n)^*\psi(n) & -\mathrm{i}\psi(n)^*\Delta\sqrt{\kappa(1+\frac{1}{4}(-1)^n\kappa\Delta+\frac{\kappa}{4}\Delta^2\psi(n)^*\psi(n))} \\ \mathrm{i}\Delta\sqrt{\kappa(1+\frac{1}{4}(-1)^n\kappa\Delta+\frac{\kappa}{4}\Delta^2\psi(n)^*\psi(n))}\psi(n) & 1+\frac{1}{4}(-1)^n\kappa\Delta+\frac{1}{2}\lambda\Delta+\frac{\kappa}{2}\Delta^2\psi(n)^*\psi(n)\end{pmatrix}.\end{aligned} \tag{1.4}$$

This model satisfies the required criteria but looks very complicated and does not, therefore, comply with our physical requirement of simplicity. Finally, in a further investigation [8], a relatively simple model was proposed, where the Lax operator was given directly by (1.4). However, the important criterion (iii) is not fulfilled and the simplification achieved is also not fully satisfactory, as is evident from the form of the Lax operator and structure of the following Hamiltonian:

$$H=-\frac{4}{3\kappa\Delta^3}\sum_n^N\left(t_n+t_n^\dagger+\frac{8-\kappa\Delta}{8-2\kappa\Delta}\right)+\left(\frac{4}{3\Delta^2}+\frac{\kappa^2}{12}\right)\sum_n^N\Delta\psi_n^\dagger\psi_n \tag{1.5}$$

where the local density t_n again has different expressions depending on whether it corresponds to even or odd sites. For odd n, it takes the form

$$\begin{aligned}t_n=(\alpha^\dagger(n+2)\alpha(n+1))^{-1}&\left\{(\alpha^\dagger(n)\alpha(n-1))^{-1}(\alpha^\dagger(n+1)\alpha(n))^{-1}(\alpha^\dagger(n+1)\sigma_3\alpha(n-1)\right\}\\ &\times(\alpha^\dagger(n+2)\alpha(n+1))^{-1}\end{aligned} \tag{1.5a}$$

with

$$\alpha(n)=\left(-\mathrm{i}\sqrt{\frac{\kappa}{2}}\Delta\psi_n^{\dagger},\sqrt{2}\left(1-\frac{\kappa\Delta}{2}-\frac{\kappa\Delta^2}{4}\psi_n^{\dagger}\psi_n\right)^{1/2}\right) \tag{1.5b}$$

and at even n sites, different, though similar, expressions hold for both t_n and $\alpha(n)$ [8]. The energy spectrum of this model obtained through the Bethe ansatz is also rather complicated and is simplified only at the continuum limit. As far as we know, until now, not many other proposals have been invoked to improve this situation [9], particularly to achieve simpler forms of local conserved quantities at the lattice level. However, a completely different approach was formulated in [4] through equivalence between NLS and spin models using quantum-space intertwiners.

Our primary aim here is to consider a quantum integrable lattice model, which at the continuum limit yields the more general AKNS system [13] and, as an allowed reduction, the NLS field model. The system considered fulfils *all* the desirable requirements of a discrete quantum system listed above and, most importantly, exhibits considerably simpler expressions for the related conserved quantities at the lattice level. Indeed, it satisfies the QYBE exactly with the same rational R-matrix (1.2) and also allows solution via the QISM, yields a local Hamiltonian and returns both the Hamiltonian and the Lax operator of the NLS field model at the continuum limit. Moreover, it has an extremely simple structure which induces an almost trivial form for the projector required for the construction of local Hamiltonians. Remarkably, this projector turns out to be field-independent and symmetric. As a further relevant feature, our model also allows a natural vector generalization at the lattice level. Finally, it admits an integrable deformation involving Tamm–Dancoff (TD)-type q-bosons. On the other hand, for achieving all these agreeable properties, one pays the price of the non-Hermitian nature of the physical observables at the lattice level. At the same time, the associated Lax operator lacks the usual $SU(2)$ ($SU(1,1)$) symmetry.

We should stress here that such Lax operators with lesser symmetries were found also to be significant in generating a large class of quantum integrable models [11].

2. The classical model

The model under scrutiny may be given through the Lax operator of the form

$$L(n|\zeta)=\begin{pmatrix}\zeta+\Delta\kappa\phi(n)\psi(n) & -\mathrm{i}\phi(n)(\kappa\Delta)^{1/2}\\ \mathrm{i}\psi(n)(\kappa\Delta)^{1/2} & 1\end{pmatrix}. \tag{2.1}$$

Its simplified structure compared to (1.3) is explicit, though due to the non-conjugacy relation between ϕ and ψ, it is obviously not Hermitian. Note that similar forms of L-operators also appear when analysing descrete self-trapping systems [14] as well as integrable systems close to the Toda lattice [15].

Recently, the bi-Hamiltonian structure of the classical system corresponding to (2.1) has been determined and its complete integrability has been rigorously established [16] through the explicit construction of action-angle variables using the r-matrix approach. Recall that, at the classical limit, the QYBE (1.1) reduces to the classical Yang–Baxter equation

$$\{L(n|\zeta)\otimes, L(m|\eta)\}=[r(\zeta,\eta), L(n|\zeta)\otimes L(m|\eta)]\delta_{nm}.$$

For the present model, the quantum R-matrix is given by (1.2) and is related to its classical counterpart r by

$$\frac{1}{\zeta}PR(\zeta) = I - \mathrm{i}\kappa r(\zeta) \qquad r(\zeta) = \frac{\Delta}{\zeta}P.$$

Now, to show the transition of the Lax operator to that of the continuum model, one should put $\zeta = 1 + \mathrm{i}\Delta\lambda$ and $\psi(n) \to \mathrm{i}\sqrt{\Delta}\psi(n)$, $\phi(n) \to -\mathrm{i}\sqrt{\Delta}\phi(n)$, which by introducing $\psi(n) = \frac{1}{\Delta}\int_{x_n}^{x_n+\Delta}\psi(x)\,\mathrm{d}x$, and a similar expression for ϕ, would yield, from equation (2.1), $L(n|\zeta) = 1 + \Delta\mathcal{L}(x,\lambda) + \mathrm{O}(\Delta^2)$.

$\mathcal{L}(x,\lambda)$ is the Lax operator of the corresponding field model, given by

$$\mathcal{L}(x,\lambda) = \begin{pmatrix} \mathrm{i}\lambda & \kappa^{1/2}\phi \\ \kappa^{12}\psi & 0 \end{pmatrix}. \tag{2.2}$$

It may be easily checked that the conserved quantities associated with this system are the same as those of the AKNS system [13]; moreover, since their Poisson structures coincide, one may conclude that the two systems are equivalent. In fact, through a simple gauge transformation

$$\mathcal{L} \to h\mathcal{L}h^{-1} + h_x h^{-1} \qquad h = \exp\left(-\mathrm{i}\frac{\lambda}{2}x\right) \tag{2.3}$$

this $\mathcal{L}$-operator can be changed into the standard Lax operator of continuum NLS

$$\mathcal{L}(x,\lambda) = \begin{pmatrix} \mathrm{i}\lambda 2 & \kappa^{1/2}\phi \\ \kappa^{1/2}\psi & -\mathrm{i}\frac{\lambda}{2} \end{pmatrix} = \mathrm{i}\frac{\lambda}{2}\sigma^3 + \kappa^{1/2}\phi\sigma^+ + \kappa^{1/2}\psi\sigma^- \tag{2.4}$$

restoring the unitary symmetry, since, as is well known, the AKNS system allows the reduction $\phi = \psi^*$.

3. Quantum model

Recently, more general forms of the L-operator of discrete quantum integrable models corresponding to standard R-matrices have been proposed [11]. Such a class of L-operators associated with the rational R-matrix (1.2) and satisfying the QYBE may be given by the following expression which clearly lacks the unitary symmetry:

$$L = \begin{pmatrix} K_1 + \mathrm{i}\frac{\lambda}{\kappa}K_2 & K_- \\ K_+ & K_3 + \mathrm{i}\frac{\lambda}{\kappa}K_4 \end{pmatrix} \tag{3.1}$$

where K-operators satisfy the algebra

$$\begin{aligned} &[K_+, K_-] = (K_1K_4 - K_2K_3) \qquad [K_1, K_3] = 0 \\ &[K_1, K_\pm] = \pm K_\pm K_2 \qquad [K_3, K_\pm] = \mp K_\pm K_4 \end{aligned} \tag{3.2}$$

with K_2, K_4 as central elements. It may be seen that when $K_1 = -K_3$, $K_2 = K_4 = 1$ and $K_+ = (K_-)^\dagger$, equation (3.2) reduces to the standard $su(2)$ algebra and one can get back the known lattice NLS (1.3) through HPT. However, the L-operator (3.1), in general, gives the possibility of generating other quantum integrable models which do not exhibit such symmetry. The quantum Toda chain is one of the main examples [11]. It is interesting to observe that the quantum version of the NLS model (2.1) considered here, also falls into this class and can be obtained from (3.1) through the following realization:

$$\begin{aligned} &K_1 = \Delta^2\kappa\phi\psi + 1 \qquad K_2 = -\Delta\kappa \qquad K_3 = 1 \qquad K_4 = 0 \\ &K_+ = \mathrm{i}\Delta\sqrt{\kappa}\psi \qquad K_- = -\mathrm{i}\Delta\sqrt{\kappa}\phi \end{aligned} \tag{3.3}$$

where the operators ψ, ϕ obey the canonical commutation relation $[\psi(n), \phi(m)] = \frac{1}{\Delta}\delta_{nm}$. This quantum model, represented by the Lax operator

$$L(n|\lambda)_{\mathrm{NLS}} = \begin{pmatrix} 1 - \mathrm{i}\lambda\Delta + \Delta^2\kappa\phi(n)\psi(n) & -\mathrm{i}\Delta\kappa^{1/2}\phi(n) \\ \mathrm{i}\Delta\kappa^{1/2}\psi(n) & 1 \end{pmatrix} \tag{3.4}$$

as a descendant of the integrable 'ancestor' model (3.1), is naturally quantum integrable and satisfies the QYBE with the same R-matrix (1.2) as the NLS field model.

In exact analogy with the classical case, equation (3.4) allows transition to the Lax operator of the AKNS system and, through allowed reduction, to that of the continuum quantum NLS model. Indeed, the gauge transformation (2.3), being independent of the field operators, is clearly applicable to the quantum case as well.

It is known [7] that the conserved quantities C_l may be obtained from the transfer matrix $\tau(\lambda) = \mathrm{tr}(\prod_N^1 L(n|\lambda))$ through an expansion at a special point ν in the form

$$C_l = \frac{1}{\kappa l!}\frac{\partial^l}{\partial\lambda^l}\log\tau(\lambda)|_{(\lambda=\nu)}.$$

In what follows, we use the method developed in [7, 8]. The locality of the Hamiltonian and other conserved quantities can be achieved provided that at this special point ν the operator $L(\lambda)$ is expressible both as a 'direct' and an 'inverse' one-dimensional projector [7, 9]. This in turn implies the vanishing of its quantum determinant [6] $\det_q L$ at this point, where

$$\begin{aligned} \det_q L &= \mathrm{tr}(\mathcal{P}_-(L(\lambda)\otimes L(\lambda+\mathrm{i}\kappa))) \\ &= \tfrac{1}{2}\left[(L_{11}\tilde{L}_{22} + L_{22}\tilde{L}_{11}) - (L_{21}\tilde{L}_{12} + L_{12}\tilde{L}_{21})\right] \end{aligned}$$

with $\mathcal{P}_- = \frac{1}{4}(1 - \sum_a \sigma_a\otimes\sigma_a)$ being the antisymmetrizer and $L \equiv L(\lambda)$, $\tilde{L} \equiv L(\lambda+\mathrm{i}\kappa)$.

We observe that for Lax operator (3.4), one gets $\det_q L = 1 - \mathrm{i}\lambda\Delta$, giving a single degeneracy point $\nu_1 = -\mathrm{i}/\Delta$. The resulting projector depends on the field operators and one cannot avoid the implementation of the involved procedure discovered and applied in [6–8] and elaborated in [9]. Fortunately, however, under an irrelevant scaling of the Lax operator $L \to \hat{L} = (\mathrm{i}/\lambda\Delta)L$, which evidently does not affect the QYBE and, thus, can only give equivalent lattice models. The quantum determinant becomes

$$\det_q \hat{L} = -\frac{1}{\Delta^2}\left(\frac{1-\mathrm{i}\lambda\Delta}{\lambda(\lambda+i\kappa)}\right) = \frac{\xi(\xi+\Delta)}{\Delta^2(1+\kappa\xi)}$$

where $\xi = \mathrm{i}/\lambda$. That is, another degeneracy point $\xi = \nu_2 = 0$ naturally appears.

The rescaled operator $\hat{L}$ takes the form

$$\hat{L}(n|\xi) = \begin{pmatrix} 1 + \dfrac{N(n)}{\Delta}\xi & -\mathrm{i}\kappa^{1/2}\phi(n)\xi \\ \mathrm{i}\kappa^{1/2}\psi(n)\xi & \dfrac{1}{\Delta}\xi \end{pmatrix} \tag{3.4$'$}$$

with $N(k) = 1 + \kappa\Delta^2\phi(k)\psi(k)$. At the new degeneracy point $\xi = \nu_2 = 0$, it becomes remarkably simple as it turns into a field-independent projector

$$\hat{L}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{P}. \tag{3.5}$$

The above procedure amounts essentially to choosing the expansion point at $\lambda = \infty$. We emphasize that the existence of such an exceptional expansion point where the projector becomes field independent is possible only due to the asymmetry of the present model. We note incidentally that an analogous property also holds at the classical level [16]. As a consequence, due to the almost trivial form of $\hat{L}(0)$ (3.5), as we will see now, not only is the required locality condition satisfied, but the derivation as well as the expression for the Hamiltonian and the other conserved quantities becomes extremely simple.

For explicit calculations we use now $\hat{L}$ and expand around $\xi = 0$ assuming periodic boundary conditions, dropping, however, the *hat* sign from all subsequent expressions. This gives

$$\tau(0) = \mathrm{tr}\left(\prod L(k|\xi)|_{(\xi=0)}\right) = 1 \tag{3.5a}$$

$$\begin{aligned} \frac{\partial}{\partial\xi}\tau(\xi)|_{\xi=0} \equiv \tau'(0) &= \mathrm{tr}\sum_k (L(N|\xi)\cdots L'(k|\xi)\cdots L(1|\xi))|_{(\xi=0)} \\ &= \frac{1}{\Delta}\,\mathrm{tr}\sum_k (\mathcal{P}\cdots N(k)\mathcal{P}\cdots\mathcal{P}) \\ &= \frac{1}{\Delta}\sum_k N(k). \end{aligned} \tag{3.5b}$$

In a similar way, one gets

$$\tau''(0) = 2\left(\frac{1}{\Delta^2}\sum_{i>k} N(i)N(k) + \kappa\sum_k \phi(k+1)\psi(k)\right) \tag{3.5c}$$

where a factor 2 appears due to the identity

$$(\cdots L(i)\cdots L'(k)\cdots)'|_{\xi=0} = (\cdots L'(i)\cdots L(k)\cdots)'|_{\xi=0}$$

and is valid since $L''(\xi) = 0$. Continuing further, we get

$$\begin{aligned} \tau'''(0) = \frac{6}{\Delta}\Bigg(\frac{1}{\Delta^2}\sum_{i>j>k} N(i)N(j)N(k) \\ + \kappa\Bigg(\sum_{i,k(i\neq k,\neq k+1)} N(i)\phi(k+1)\psi(k) + \sum_k \phi(k+1)\psi(k-1)\Bigg)\Bigg). \end{aligned} \tag{3.5d}$$

Notice that the conserved quantities C_k may be given through the above expressions (3.5) in the following form:

$$C_1 = \frac{1}{\kappa}(\log \tau(\xi))'|(\xi = 0) = \frac{1}{\kappa}\tau(0)^{-1}\tau'(0)$$

$$C_2 = \frac{1}{2\kappa}(\log \tau(\xi))''|(\xi = 0) = \frac{1}{2\kappa}\left[\tau(0)^{-1}\tau''(0) - (\tau(0)^{-1}\tau'(0))^2\right]$$

$$C_3 = \frac{1}{3!\kappa}(\log \tau(\xi))'''|(\xi = 0) = \frac{1}{6\kappa}\left[2(\tau^{-1}\tau'(0))^3 + \tau^{-1}\tau'''(0) - 2(\tau^{-1}\tau'(0))\right.$$
$$\left.\times\,(\tau^{-1}\tau''(0)) - (\tau^{-1}\tau''(0))(\tau^{-1}\tau'(0))\right]$$

where $\tau^{-1} = \tau^{-1}(0)$. Inserting now the expressions (3.5), one finally obtains the required observables

$$N = C_1 = \frac{1}{\Delta\kappa}\sum_k N(k) \tag{3.6a}$$

as the 'number' operator,

$$P = C_2 \equiv \sum_k p_k = \sum_k \left(\phi(k+1)\psi(k) - \frac{1}{2\kappa\Delta^2}N(k)^2\right) \tag{3.6b}$$

as the 'momentum' operator and

$$H = C_3 \equiv \sum_k h_k = \frac{1}{\Delta}\sum_k \{\phi(k+1)\psi(k-1)$$
$$- [N(k) + N(k+1)]\phi(k+1)\psi(k) + (3\kappa\Delta^2)^{-1}N(k)^3\}. \tag{3.6c}$$

as the Hamiltonian of the system.

It may be noted that the above conserved quantities are not symmetric in ϕ and ψ, which is a consequence of the asymmetry of the Lax operator. On the other hand, their locality is explicit and it is interesting to observe that even though expressions (3.5), given through expansion of $\tau(\xi)$, are all non-local, in the corresponding conserved quantities (3.6), all such non-local terms cancel among themselves leaving only the local terms, as occurs also in the classical case. We stress again that the evident simplicity of expressions (3.6*a*–*c*) for the conserved quantities is the most prominent feature of the present model.

The transition of these conserved quantities to those of the NLS field model is easily achieved at the continuum limit by taking

$$\boldsymbol{N} = \left(\frac{1}{\Delta\kappa}\sum_k (N(k) - 1)\right)\Bigg|_{(\Delta\to 0)} = \int \mathrm{d}x\, \phi(x)\psi(x) \tag{3.7a}$$

$$\boldsymbol{P} = 2\left(\sum_k \left(p_k + \frac{1}{2\kappa\Delta^2}\right)\right)\Bigg|_{(\Delta\to 0)} = \int \mathrm{d}x\, (\phi_x\psi - \phi\psi_x) \tag{3.7b}$$

$$\boldsymbol{H} = -\left(\sum_k \left(h_k - \frac{1}{3\kappa\Delta^2}\right)\right)\Bigg|_{(\Delta\to 0)} = \int \mathrm{d}x\, (\phi_x\psi_x + \kappa(\phi\psi)^2) \tag{3.7c}$$

with the standard assumption of a vanishing boundary condition. It is worth remarking that the continuous conserved quantities (3.7) of the AKNS-type system are now symmetric in ϕ and ψ, which allows, therefore, the reduction $\phi = \psi^\dagger$, yielding the known expressions for the NLS field model.

The evident closeness between the conserved quantities (3.6) of the lattice version with those of (3.7) related to the continuum model is a noticeable feature of the present model. For solving the eigenvalue problem for the Hamiltonian of the discrete model exactly, we go along the well established steps [8] of the algebraic Bethe ansatz, which forms the basic tool of the QISM [10]. Defining the monodromy matrix as

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \prod_k^N L(k|\lambda)$$

we get the expression for the transfer matrix as $\tau(\lambda) = \mathrm{tr}(T(\lambda)) = A(\lambda) + D(\lambda)$, which generates the conserved quantities, while $B(\lambda), C(\lambda)$ act as 'creation' and 'annihilation' operators, respectively. The n-particle eigenstates may be defined as $|n\rangle = \prod_i^n B(\lambda_i)|0\rangle$ with the 'vacuum' properties

$$C(\lambda)|0\rangle = 0 \qquad A(\lambda)|0\rangle = a(\lambda)^N|0\rangle \qquad D(\lambda)|0\rangle = d(\lambda)^N|0\rangle.$$

The QYBE for the monodromy matrix is given again by equation (1.1) with L-operators replaced by the corresponding T-operators. In elementwise form, this equation yields the 'commutation' relations

$$\begin{aligned} &[A(\lambda), A(\mu)] = [D(\lambda), D(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\lambda)] = 0 \\ &A(\lambda)B(\mu) = \frac{1}{c(\mu,\lambda)}B(\mu)A(\lambda) - \frac{b(\mu,\lambda)}{c(\mu,\lambda)}B(\lambda)A(\mu) \\ &D(\lambda)B(\mu) = \frac{1}{c(\lambda,\mu)}B(\mu)D(\lambda) - \frac{b(\lambda,\mu)}{c(\lambda,\mu)}B(\lambda)D(\mu) \end{aligned} \tag{3.8}$$

with

$$b(\lambda,\mu) = \frac{\mathrm{i}\kappa}{\lambda-\mu-\mathrm{i}\kappa} \qquad c(\lambda,\mu) = \frac{\lambda-\mu}{\lambda-\mu-\mathrm{i}\kappa}.$$

The eigenvalues of $\tau(\lambda)$ giving the physical observables may be obtained by using the commutation relations (3.8) between A, B and D, B and the properties of the 'vacuum' stated above. Skipping out the details, we present only the main results as follows

$$\tau(\lambda)|n\rangle = \mathcal{E}(\lambda, \{\lambda_j\})|n\rangle \tag{3.9}$$

where

$$\mathcal{E}(\lambda, \{\lambda_j\}) = a(\lambda)^N \prod_l^n \frac{1}{c(\lambda_l,\lambda)} + d(\lambda)^N \prod_l^n \frac{1}{c(\lambda,\lambda_l)}. \tag{3.10}$$

Note that the form of eigenvalues (3.10) is obtained providing the parameters λ_i satisfy the condition [10]

$$\left(\frac{a(\lambda_j)}{d(\lambda_j)}\right)^N = \prod_{l\neq j}^n \frac{c(\lambda_l,\lambda_j)}{c(\lambda_j,\lambda_l)} \qquad i,j = 1,\ldots,n. \tag{3.11}$$

For the present case, we obtain $a(\xi) = (1 + \frac{\xi}{\Delta})$, $d(\xi) = \frac{\xi}{\Delta}$. We define the Hamiltonians by

$$C_k = \frac{1}{n!\kappa} \frac{\partial}{\partial \xi^k} \log(\tau(\xi) a^{-N}(\xi))|_{(\xi=0)}$$

where a^{-N} is included to remove irrelevant constant terms and to avoid linear combinations of conserved quantities. Thus, we get from (3.10)

$$C_k = \frac{1}{n!\kappa} \frac{\partial}{\partial \xi^k} \log \tilde{\tau}(\xi)|_{(\xi=0)}$$

where

$$\tilde{\tau}(\xi) = \prod_k^n \frac{1 + \mathrm{i}(\lambda_k - \mathrm{i}\kappa)\xi}{1 + \mathrm{i}\lambda_k \xi} \tag{3.12}$$

which finally yields

$$\begin{aligned} \mathcal{N} &= -C_1 = n, \\ \mathcal{P} &= -\mathrm{i}\left(C_2 + \frac{\kappa}{2} C_1\right) = \sum_{k=1}^n \lambda_k \\ \mathcal{E} &= C_3 - \kappa C_2 + \frac{\kappa^2}{6} C_1 = \sum_{k=1}^n \lambda_k^2. \end{aligned} \tag{3.13}$$

We observe that the energy is proportional to λ_k^2, the momentum is proportional to λ_k and the number of particles is equal to the quasiparticle excitation number, as required. Note again that this result concerning the descrete model under consideration is similar to that of the NLS field model [10] including the combinations of different conserved quantities to determine the momentum and energy spectra. However, contrary to the continuum case, here the values of the λ_k's are not arbitrary and should be determined from equations (3.11), which for the present model are

$$(1 - \mathrm{i}\lambda_j \Delta)^N = \prod_{l \neq j}^n \frac{\lambda_j - \lambda_k - \mathrm{i}\kappa}{\lambda_j - \lambda_k + \mathrm{i}\kappa} \qquad i, j = 1, \ldots, n. \tag{3.14}$$

We should emphasise that the energy spectrum of this model obtained above and the related constraints on λ_k are, indeed, extremely simple.

4. Vector generalization of the model

It is interesting to observe that the models violating the $SU(2)$-type symmetry, proposed in [11], can easily be generalized for the $gl(N)$ case. Out of such generalized systems, one might then construct quantum integrable models, like the multi-component Toda chain, vector NLS etc, as realizations through a set of independent bosonic operators. Such generalized systems are given by the Lax operator

$$L = \sum_l \left(K_l^+ + \frac{\mathrm{i}\lambda}{\kappa} K_l^-\right) e_{ll} + \sum_{j \neq l} K_{lj} e_{jl} \tag{4.1}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ are the generators of $gl(N)$. It can be shown that the above L-operator, associated with the rational $(N^2 \times N^2)$ R-matrix ($R(\lambda) = 1 + \mathrm{i}\frac{\lambda}{\kappa}\Pi$ where $\Pi = \sum_{lk} e_{kl} \otimes e_{lk}$), satisfies the QYBE if the generators K yield the following algebra:

$$
\begin{aligned}
&[K_{mk}, K_{kl}] = K_k^- K_{ml} \qquad (k \neq l \neq m) \qquad [K_{kl}, K_{lk}] = K_k^+ K_l^- - K_k^- K_l^+ \qquad (k \neq l) \\
&[K_k^+, K_{kl}] = K_{kl} K_k^- \qquad (k \neq l) \qquad [K_k^+, K_{lk}] = -K_{lk} K_k^- \qquad (k \neq l) \\
&[K_k^+, K_{lm}] = [K_{kl}, K_{km}] = [K_{kl}, K_{ml}] = [K_{kl}, K_{mn}] = 0 \qquad (k \neq l \neq m)
\end{aligned}
\tag{4.2}
$$

where K_k^- commute with all other generators: $[K_k^-, K_{ij}] = 0$ and thus are central elements, while $K_l^{\pm}$ form an Abelian subalgebra. We may notice again that, in general, this is not a $su(N)$ algebra, which, however, is recovered at some particular symmetric reduction.

Different realizations of this algebra would generate, through (4.1), different quantum integrable models, which would share the same rational R-matrix but generically would not exhibit unitary symmetry. Consider now a realization of (4.2) through a set of independent operators with the commutation relations $[\psi_l, \phi_k] = \delta_{lk}$ and $[\psi_l, \psi_k] = [\phi_l, \phi_k] = 0$ in the form

$$
\begin{aligned}
&K_1^- = -1 \qquad K_1^+ = \sum_j \phi_j \psi_j \qquad K_i^+ = 1_{ii} \qquad K_i^- = 0 \qquad (i = 2, \ldots, N) \\
&K_{1j} = \psi_j \qquad K_{j1} = \phi_j \qquad K_{ij} = 0 \qquad 1 < (i, j) \leqslant N.
\end{aligned}
\tag{4.3}
$$

The corresponding Lax operator (4.1) will then read

$$
L(\lambda) = \begin{pmatrix} -\mathrm{i}\lambda/\kappa + (\phi\psi) & \phi \\ \psi & \mathbf{1} \end{pmatrix}. \tag{4.4}
$$

This Lax operator, which yields a quantum integrable lattice model, gives the vector NLS model [17, 18] at the continuum limit and is a natural generalization of (3.4) to the vector case. The associated R-matrix also coincides with that of the field model [18]. Thus, (4.4) is related to the Lax operator of a new exactly integrable lattice version of the vector NLS model. The corresponding classical system has been considered in [16].

5. A novel quantum integrable Tamm–Dancoff q-bosonic model

A number of lattice models involving q-oscillators, which are integrable at the quantum level, have already been discovered [11, 19]. Most of these models are related to the quantum group structures associated with the trigonometric R-matrix, which forms a separate class entirely different from the NLS model with rational R-matrix (1.2). We present here an integrable deformation of the discrete NLS model (3.4), which involves Tamm–Dancoff (TD)-type q-boson operators [20] but at the same time is related to a rational R-matix.

It has been shown in [11] that for a 'symmetry breaking' transformation [21], $R_{ij}^{kl}(\lambda) \rightarrow \mathrm{e}^{\mathrm{i}\theta(j-k)} R_{ij}^{kl}(\lambda)$ of the original R-matrix (1.2) where θ is some constant parameter. The algebra (3.1) of K operators is also deformed in an interesting way. We find a realization of this deformed algebra through Tamm–Dancoff (TD)-type q-bosonic operators b, c, N

$$
[b, N] = b \qquad [c, N] = -c \qquad bc - qcb = q^N \tag{5.1}
$$

where b and c are not, in general, Hermitian conjugates of each other. Here we have introduced the parameter $q = e^{i2\theta}$. One may compare the above TD-type q-deformed bosons with the standard q-oscillator algebra [22]: $[a, N] = a$, $[a^\dagger, N] = -a^\dagger$, $aa^\dagger - qa^\dagger a = q^{-N}$. The algebraic relations (5.1) yield the Lax operator of the corresponding model as

$$L^q(\lambda) = \begin{pmatrix} (1 + \kappa N - i\lambda) f(N) & -i\kappa^{1/2} c \\ i\kappa^{1/2} b & f(N) \end{pmatrix} \tag{5.2}$$

where $f(N) = q^{\frac{1}{2}(N-\frac{1}{2})}$.

Note that it represents a quantum integrable system, which satisfies the QYBE with the deformed rational R-matrix

$$R^q(\lambda) = \begin{pmatrix} \lambda - i\kappa & & & \\ & -i\kappa & \lambda q^{-1/2} & \\ & \lambda q^{1/2} & -i\kappa & \\ & & & \lambda - i\kappa \end{pmatrix}. \tag{5.3}$$

Evidently, at $q \to 1$, one recovers from (5.3) the Lax operator (3.4) of our discrete NLS and also obtain $R^q \to R$ as in (1.2).

There is a simple mapping from such TD-deformed operators to the operators of the original lattice NLS model as $b = f(N)\psi$ and $c = f(N)\phi$, recovering the canonical relation $[\psi, \phi] = 1$; accordingly, the Lax operator (5.2) is mapped into (3.4) by

$$L^q(n) = q^{\frac{1}{2}(N(n)-\frac{1}{2})} L(n)_{\mathrm{NLS}}.$$

Hence, this TD-type deformed bosonic system represents a new quantum integrable lattice model which can be solved through the algebraic Bethe ansatz using the results reported in section 3.

6. Concluding remarks

After the completion of this work, references [23, 24] were brought to our notice. In [23], as a fundamental contribution, a most general form of L-operator for the lattice NLS was found, which provides the basis for classification of all L-operators related to the R-matrix (1.2). The physical and mathematical properties of lattice NLS along with many other models have been discussed in great detail in [24]. We have checked that the general L-operator of [23] may be represented by (3.1) for a particular realization of algebra (3.2) through $(\psi^\dagger(n), \psi(n))$ with $[\psi^\dagger(n), \psi(n)] = 1$ as

$$K_1 = i\kappa a_n^{(1)} \psi^\dagger(n)\psi(n) + a_n^{(0)} \qquad K_2 = -i\kappa a_n^{(1)} K_3 = -i\kappa d_n^{(1)} \psi^\dagger(n)\psi(n) + d_n^{(0)}$$

$$K_4 = -i\kappa d_n^{(1)} \qquad K_+ = \kappa\rho(n)\psi(n) \qquad K_- = \psi^\dagger(n)$$

where

$$\rho(n) = \kappa a_n^{(1)} d_n^{(1)} \psi^\dagger(n)\psi(n) + i(a_n^{(1)} d_n^{(0)} - d_n^{(1)} a_n^{(0)}).$$

Since the Lax operator (3.4) of our lattice model is also a special case of (3.1), it is naturally consistent with the form (6.1) related to the L-operator of [23], which, thus, gives another basis for its validity.

A few more comments are in order at this point, to stress the different motivations underlying our paper with respect to [23]. In fact, the basic goal pursued and brilliantly achieved in [23] was to find a family of L-operators which generate all the monodromy matrices related to the XXX R-matrix, and consequently no concrete model construction was undertaken. Our aim was instead the construction of a *concrete* integrable quantum model, arising from a *concrete* integrable classical model, thoroughly investigated in [16]. It was again the study of the classical model and the possibility of constructing an integrable vector generalization at the classical level that paved the way for introducing algebra (4.2) and the vector generalization at the quantum level, which was not considered in [23].

To conclude, we should say that the main original contributions contained in our paper are as follows.

(1) The explicit derivation of the conserved quantities, including the Hamiltonian, as well as their spectra, for a quantum integrable lattice model (already introduced in [23]), which, in general, corresponds to the AKNS-type system and may also be considered as a lattice NLS model.

(2) The construction of an integrable vector generalization of the model that provides a new exact lattice version of the vector NLS field model which is to our knowledge much simpler than all other vector generalizations available in the literature.

(3) The construction of a simple deformed model, i.e. a TD-type q-boson model, exactly solvable at the quantum level.

The price we have had to pay to achieve all the previous results is the breaking of unitary symmetry, which is, however, restored at the continuum limit. We stress again that the same advantages and drawbacks also characterize the classical version investigated in [16].

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